

Soft-Gluon Cancellation, Phases and Factorization with Initial-State Partons

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We outline arguments for the cancellation of soft singularities in transition probabilities and parton distributions with incoming partons and Wilson lines, and with observed jets or heavy colored particles in the final states. We show the cancellation of Glauber gluons and divergent phases, relating finite remainders to corrections that arise from restrictions on final states in factorized cross sections.

In this paper we investigate the cancellation of soft-gluon singularities for inclusive transition probabilities with a parton and a Wilson line in the initial state. Our arguments apply to a large class of parton distributions, and outline an extension of the proof of factorization in Ref. [1] for Drell-Yan and related processes [2, 3, 4] to cross sections with colored particles in the initial state and with final-state jets. This requires us to explore the role of “Glauber” or “Coulomb” regions of momentum space [2, 5, 6, 7] in these processes. These regions, where soft gluons carry unphysical spacelike momenta, are related to the generation of phases [8, 9].

Transition Probabilities. We consider weighted transition probabilities with an initial-state Wilson line in color representation f' and a parton of flavor f and momentum p ,

$$\mathcal{S}_f[S] = \sum_N S(N) \text{Tr}_c \left[\langle p | \bar{T} \left[\left(\phi_f(0) \Phi_{\beta}^{(f')}(0, -\infty) \right)^{\dagger} \right] | N \rangle \right. \\ \left. \times \langle N | T \left(\phi_f^{\dagger}(0) \Phi_{\beta}^{(f')}(0, -\infty) \right) | p \rangle \right], \quad (1)$$

where T and \bar{T} denote time- and anti-time-orderings and $f \equiv \{f, f'\}$. The field of the incoming parton is ϕ_f , $\beta^{\mu} = \delta_{\mu+}$ is a light-like four-velocity in the plus direction, and $\Phi_{\beta}^{(f')}(b, a) = P \exp \left[-ig \int_a^b d\lambda A_{\beta}(\lambda\beta) \right]$ is the Wilson line, with f' the group representation for the gauge field $A_{\beta} \equiv \beta \cdot A$. We take $p^{\mu} = p^{-} \delta_{\mu-}$ opposite to β . The function $S(N)$ assigns weights to final states $|N\rangle$. For our arguments we will assume that $S(N)$ satisfies the usual requirements for infrared safety [10], and depends implicitly on a hard scale Q . It may be chosen, for example, to select final states with heavy particles and/or jets with specified energies and directions. The weight function $S(N)$ may be also chosen to fix the total transverse momentum of the final state. The connection between jet and single-particle inclusive cross sections with hadronic initial states was recently reviewed in Ref. [11]. Defining ϕ_f with an appropriate spin projection, the class of probabilities \mathcal{S}_f includes a variety of gauge-invariant [12] spin-dependent parton distributions with incoming Wilson lines [7, 13]. The all-orders cancellation of soft singularities in such distributions is to our knowledge a new result. The direct relation to factorization in hadron-hadron scattering is through the “weak factorization” of

Refs. [2, 4].

Cancellation for the Target Field. Expanding the Wilson lines of (1) and writing the expansion in terms of momentum space integrals of Fourier-transformed fields gives

$$\mathcal{S}_f[S] = \sum_{n=0}^{\infty} g^n \prod_{i=1}^n \int d^4 q_i \sum_{j=1}^n E^{(f')n}_j(\beta \cdot q_i) \\ \times \sum_N S(N) \text{Tr}_c \left[\langle p | A_{\beta}(q_{j+1}) \times \cdots \times A_{\beta}(q_n) \phi_f^{\dagger}(0) | N \rangle \right. \\ \left. \times \langle N | \phi_f(0) A_{\beta}(q_1) \times \cdots \times A_{\beta}(q_j) | p \rangle \right] \\ \equiv \sum_{n,j} E^{(f')n}_j \otimes_q \sum_N R_f^{(N,n,j)}[S], \quad (2)$$

where q_i is the momentum flowing out of field $A(\beta\lambda_i)$, and all dependence on final states is in $R_f^{(N,n,j)}$. In covariant gauges, soft-gluon singularities arise only from regions where sets of the q_i vanish [14]. The position-space integrals in the ordered exponentials of Eq. (1) give eikonal $\beta \cdot q$ -dependent factors, independent of final state N ,

$$E^{(f')n}_j(q_1^-, \dots, q_m^-) = \left(\frac{1}{q_{j+1}^- + i\epsilon} \frac{1}{q_{j+1}^- + q_{j+2}^- + i\epsilon} \cdots \right. \\ \left. \times \frac{1}{q_{j+1}^- + \cdots + q_n^- + i\epsilon} \right) (-1)^{n-j} \\ \times \left(\frac{1}{q_1^- + \cdots + q_j^- + i\epsilon} \cdots \frac{1}{q_{j-1}^- + q_j^- + i\epsilon} \frac{1}{q_j^- + i\epsilon} \right), \quad (3)$$

where $q^- \equiv \beta \cdot q$. The $E^{(f')n}_j$ satisfy [4, 15]

$$\sum_{j=0}^n E^{(f')n}_j(q_1^-, \dots, q_n^-) = 0, \quad (4)$$

a result we will refer to as the “eikonal identity” below. First, however, we must discuss the functions R , recalling the analysis of Ref. [1].

In Ref. [1] the remainder functions $R_f^{(N,n,j)}$ were treated in light cone-ordered perturbation theory (LCOPT), by integrating over the plus momenta of their internal loops at fixed minus and transverse momenta, including an integral over the total plus momentum flowing

from the Wilson line into the final state, p_N^+ . In this formalism, the cancellation of final state interactions [6] is manifest after the integral over p_N^+ , and one finds that for fixed n , the result is actually independent of j , that is, of how the set of gluons is split between the left and right eikonals,

$$\sum_N R_f^{(N,n,j)} = \mathcal{P}^n[S, q_i^-]. \quad (5)$$

Corrections vanish for a fully inclusive sum, $S(N) \equiv 1$, and are otherwise infrared finite. The function \mathcal{P}^n is given by purely initial-state LCOPT factors,

$$\begin{aligned} \mathcal{P}^n[S, q_i^-] &= \sum_N S(N) \sum_{T \in \mathcal{T}_I} \int_{k_l} I_T^{(N)'}(k_l) I_T^{(N)}(k_l) \\ &\quad \times \prod_{i=1}^n \theta\left(\delta_i^{(T)} q_i^-\right), \end{aligned} \quad (6)$$

where the sum is over those orderings T in which every vertex precedes the annihilation of the incoming parton, and where the integrals are over internal minus and transverse loop momenta, k_l . For each light-cone ordering T , $\delta_i^{(T)} = \pm 1$ fixes the flow of q_i^- from past to future, independent of j . The functions I_T are products of LCOPT denominators, given by the plus momentum deficit of each initial state, ξ , relative to the incoming state with $p^+ = 0$,

$$I_T^{(N)}(k_l) = \prod_{\xi < H} \frac{1}{-s_\xi + i\epsilon}, \quad s_\xi \equiv \sum_{\text{lines } j} \frac{k_{j,\perp}^2}{2k_j^-}, \quad (7)$$

where $<$ refers to earlier in light-cone ordering, and where we suppress numerator factors. We denote as H the vertex at which the parton f is annihilated, to divide initial from final states. Using Eqs. (4) and (5) in Eq. (2), we conclude that, when all the q_i^- are nonzero, the sum over all connections of gluons to the initial-state Wilson line vanishes, and soft-gluon singularities cancel.

The Eikonal Field and Induction. By itself, the forgoing would appear to eliminate all gluon attachments to the eikonal lines and rule out all soft-gluon singularities in Eq. (1). This argument has an important limitation, however, related to the output of LCOPT, Eq. (6). This is an ambiguity in the integrations at points $q_i^- = 0$ where, in general, the eikonal factors are singular. On the one hand, the eikonal singularity at $q^- = 0$ for a virtual gluon exchanged between the incoming Wilson line and the incoming parton f is clearly related to a divergent imaginary “phase”. On the other hand, in order to use the eikonal identity (4), we must sum over final states with gluons whose plus momenta are given by $q_i^+ = q_{i,T}^2/2q_i^-$. In any realistic case, there is a maximum value, Q , on the q_i^+ and hence a lower limit on q_i^- when q_i appears in the final state. We shall refer to partons whose minus components are large enough to satisfy this bound the *target field*. The cancellation of imaginary

parts is evidently not included in the eikonal identity of Eq. (4), which applies to the target field only.

In summary, to establish cancellation we must extend the analysis of Ref. [1] to regions of arbitrarily low q^- . We will refer to the set of low- q^- lines connected to the incoming eikonal line as the *eikonal field subdiagram*, $\mathcal{E}^{(f')}$. In the eikonal field subdiagram, the plus momentum integrals that lead to LCOPT expansions do not generally converge [1], which leads precisely to the ambiguity in the minus integrals noted above. We shall treat this problem below.

In general, lines of the eikonal field subdiagram $\mathcal{E}^{(f')}$ may appear in the final state carrying plus momenta of order Q . Such lines are “spectator lines” of a jet in the β direction. Adapting the reasoning of Ref. [1], one can show that β jet spectators factorize from soft gluons after a sum over final states, and hence do not participate in soft divergences. We will assume this result here, and review details of the extended argument elsewhere [16]. In this paper, we will show cancellation for transition probability $\mathcal{S}_f^{(\mathcal{E})}$, in which the eikonal field subdiagram is entirely virtual. Our argument for $\mathcal{S}_f^{(\mathcal{E})}$ is inductive, with the cancellation for the pure target field, described above, as the inductive basis. Induction is carried out in the loop order of the (virtual) eikonal field subdiagram, $\mathcal{E}^{(f')}$. In this transition probability, we must sum over only those states for which $\mathcal{E}^{(f')} = \mathcal{E}_{M^*}^{(f')} \oplus \mathcal{E}_M^{(f')}$, a sum of independent subdiagrams of the amplitude and its complex conjugate. The on-shell plus momenta of all final state particles is then bounded by the hard scale Q . We denote these “target field” final states by N_T .

An IR safe weight function $S(N)$ can have only smooth dependence on target field states, and for most of the remaining analysis we set $S(N) = 1$, and show how infrared divergences associated with these final states cancel. We will then return to the choice of $S(N)$. In the discussion below, we will also restrict ourselves to regions in momentum space where all transverse momenta are of a comparable size, which we denote by m . We will see below that extensions to multiple scales are intimately connected with the choice of the weight $S(N)$. For now, m^2/Q represents a lower limit for the minus momenta in the target field.

The Hybrid Form. To discuss the role of the eikonal field, we express contributions to the transition amplitude for fixed eikonal connections n and j in Eq. (1), summed over target field final states N_T , as

$$\begin{aligned} \sum_{N_T} E^{(f')}_j^n \otimes R_f^{(N_T,n,j)}[1, q_i^-] &= \\ \sum_{N_T} \int d\tau_{N_T} M_{n-j}^*(p_t) M_j(p_t), \end{aligned} \quad (8)$$

where p_t denotes the momenta of lines t in the final state, and $d\tau_{N_T}$ their phase space measure. Once again, final state interactions cancel in the sum over N_T .

Consider now a graphical contribution to the amplitude $M_j(p_t)$ in Eq. (8). In general, it is a combination

of an eikonal subdiagram $\mathcal{E}_M^{(f')}$ and a “reduced remainder”, $r_{\mathcal{E}^{(f')}} \equiv M_j/\mathcal{E}_M^{(f')}$, none of whose lines have minus momenta of order m^2/Q or less. In general, the plus momenta of lines q_i of $\mathcal{E}_M^{(f')}$ that attach to $r_{\mathcal{E}^{(f')}}$ are “pinched” between poles at $q_i^+ \sim \pm(m^2 - i\epsilon)/Q$ from active and spectator lines of $r_{\mathcal{E}^{(f')}}$ in covariant perturbation theory [2, 5]. Such lines are often referred to as “Glauber exchanges”. We now evaluate $r_{\mathcal{E}^{(f')}}$ in LCOPT, treating the eikonal field as an external source of gluons at fixed momenta, q_i^+ .

LCOPT provides a sum of x^- -ordered diagrams for $r_{\mathcal{E}^{(f')}}$. Suppressing numerator momenta, the amplitudes are, as above, products of inverse plus momentum deficits between the total plus momentum that has flowed into the diagram before each state, i , and the on-shell plus momentum, s_i of all the particles in the state. We refer to the resulting expression as a *hybrid* form for M_j ,

$$M_j(p_i) = \int_q \mathcal{E}_M^{(f')}(\{q_a, q_b\}) \prod_{i < H} \frac{1}{-\sum_{a \in i} q_a^+ - s_i + i\epsilon} \times \prod_{j > H} \frac{1}{\sum_{b \in j} q_b^+ + k_{N_T}^+ - s_j + i\epsilon}. \quad (9)$$

The first (second) product of LCOPT denominators represents initial (final) states relative to vertex H . The function $\mathcal{E}_M^{(f')}(\{q_a, q_b\})$ denotes the eikonal field subdiagram, whose low- q^- external lines attach to $r_{\mathcal{E}^{(f')}}$ either in initial states (q_a) or final states (q_b), with an integral over the allowed region for all loop momenta. The s_i are sums of terms of order m^2/Q for particles in the p jet, and order m for soft particles. On-shell plus momenta can be order Q for particles in outgoing jets, but $k_{N_T}^+ - s_j$ is order m or less.

Glauber Exchange and the Causal Identity. By construction, all lines of $\mathcal{E}^{(f')}$ have minus momenta of order m^2/Q and transverse momenta of order m ; hence singularities of the $q_{a,b}^+$ from $\mathcal{E}^{(f')}$ in Eq. (9) are at order Q . In addition, in Eq. (9), the $q_{a,b}^+$ encounter poles from $r_{\mathcal{E}^{(f')}}$ only in the UHP when they enter $r_{\mathcal{E}^{(f')}}$ in an initial state, and only in the LHP for a final state. As a result, we may deform the $q_{a,b}^+$ contours to order Q from the origin to complex, collinear- β values. This deformation is into the LHP for the q_a , flowing out of initial states, and into the UHP for the q_b and final states. Along the deformed contours, $k_{N_T}^+ - s_j$ can be neglected in each term individually. The Glauber “pinch” of covariant perturbation theory is then replaced in the hybrid form by a mismatch between the directions of the deformations for initial and final states. In addition, we see that the eikonal field gluons can appear only with one denominator per integration if the result is to be leading power in Q . The only orderings that survive are those in which, say, h eikonal field gluons appear in all possible combinations in l initial and $h-l$ final states. Each such integral has logarithmic power counting overall.

We now distinguish between “active-connected” eikonal field gluons that attach to “active” lines in

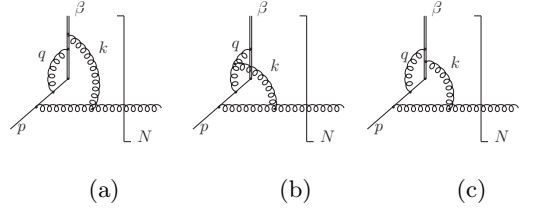


FIG. 1: Examples for active and spectator connected diagrams in LCOPT.

$r_{\mathcal{E}^{(f')}}^{(f')}$, whose minus momenta flow into vertex H , and “spectator-connected” lines, that attach to lines in $r_{\mathcal{E}^{(f')}}$ whose minus momenta flow into the final state. Examples are shown in Fig. 1. When active-connected eikonal field lines acquire large real plus momenta, they are properly considered part of the β jet, and appropriate subtractions will ensure convergence of the corresponding q_a^+ integrals [4]. Spectator lines with large plus momentum in the hybrid form are not naturally part of the β jet; indeed they are a reexpression of Glauber exchange. No subtraction is necessary, however, because the sums over spectator attachments to initial and final states cancel by the same identity as in Eq. (4),

$$\sum_{a=0}^n \prod_{k=1}^a \frac{1}{-\sum_{i=1}^k q_i^+} \prod_{l=a+1}^n \frac{1}{\sum_{j=l}^n q_j^+} = 0, \quad (10)$$

which can be interpreted as the quantum-mechanical suppression of acausal correlations [16]. Note that the “causal identity” (10) holds for fixed color orderings in each diagram, and whether or not observed jets are included in the final state. The contributions of large-plus momentum gluons attached to spectators, although present in individual LCOPT diagrams, thus vanish in the sum.

In summary, once it is collinear-subtracted, M_j will have no contributions from large plus momenta. At the same time, we cannot simply ignore the spectator-connected eikonal field, because the use of the causal identity assumes that we deform the q_b^+ and q_a^+ contours into *the same half-plane*. This requires us to cross poles from the LCOPT denominators in (9), setting some q_b^+ , for example, to values characteristic of the intermediate states of the target field, of order $\leq m$. Similar considerations apply to active-connected eikonal gluons, because the subtractions for the β jet should be defined with respect to an outgoing rather than incoming Wilson line [4]. The poles, then, are the remaining effects of the divergent plus integrals of the eikonal field, and are the origin of divergent phases not present in the purely target field integrals.

Wilson Lines and Spectator Interactions. To treat the pole contributions, we sum over the amplitudes M_j for fixed final states, N_T . As we have seen, the eikonal field momenta $q_{a,b}$ in (9) appear only in a minimum number of initial or final states. As a result, the number of spectators is conserved among these states. The initial- (final-) state interactions of eikonal

field gluons with spectators then factorize from the spectator lines onto incoming (outgoing) Wilson lines of the corresponding color representations (labelled t below), [1, 4, 17]. Nonvanishing eikonal-field-spectator interactions are thus generated by the operators [16]

$$W_-^{(t)}(\mathbf{x}_t) = \Phi^{(t)}(\infty, \mathbf{x}_t) \Phi^{(t)}(\mathbf{x}_t, -\infty). \quad (11)$$

These Wilson lines are local with respect to transverse momenta, which flow into the initial states of M_j , Eq. (9). We may then reorganize the sum $\mathcal{M} \equiv \sum_j M_j$ as

$$\mathcal{M}(p_a, \{p_t\}) = \left\langle 0 \left| \Phi^{(f')}(0, -\infty) C_{f'f}(p_a) \Phi^{(f)}(0, -\infty) \right. \right. \\ \left. \times \prod_t \int d^2 \mathbf{x}_t e^{i \mathbf{p}_{t,\perp} \cdot \mathbf{x}_t} W_-^{(t)}(\mathbf{x}_t) \right| 0 \rangle \Psi_f(\{p^-_t, \mathbf{x}_t\}), \quad (12)$$

with $C_{f'f}(p_a)$ a short-distance function that links the incoming Wilson lines in representations f' and f . The vertex $C_{f'f}(p_a)$ includes all information on the production of jets or heavy colored particles, whose interactions cancel in the sum over final states. In general, $C_{f'f}(p_a)$ is a sum over terms in all color representations that arise from the direct product of the two partonic representations of f' and f . It depends on β and on the total “active parton” momentum p_a , collinear to p , that takes part in the hard scattering. Because soft interactions between final-state jets and other particles cancel [15], the color representation in $C_{f'f}(p_a)$ and in the corresponding factor in the complex conjugate amplitude is the same for an inclusive cross section.

In Eq. (12), Ψ_f is a light-cone wave function [18] for the incoming parton p , which incorporates all initial-state dependence from states earlier than the earliest connection of the eikonal field to the target field. The wave function depends on p and a set of collinear momenta p_t^- of the observed spectators, each at a fixed transverse position, \mathbf{x}_t . Of course, $p = p_a + \sum_t p_t$. In effect, we find that at the level of amplitudes, Glauber gluons act to dress light-cone wave functions with nonabelian phases.

We now treat \mathcal{M} in hybrid form. The hard-scattering vertex and the “midpoints” of the spectator eikonals $W_-^{(t)}$ at $x = \mathbf{x}_t$ are local in x^- , so that plus momentum loops flow from the eikonals to the hard scattering in LCOPT at this vertex, just as in Eq. (9). Now at fixed plus momenta $\leq Q$, the only enhancements for minus momenta $\leq m^2/Q$ are from the eikonal denominators. Thus, loops of $\mathcal{E}^{(f')}$ that do not flow directly through the β eikonal must carry plus momenta of order Q to give leading contributions, as for momentum k in the example of Fig. 1b. This means that all loops of any eikonal field subdiagram that connects the β eikonal to more than one spectator or to the active line and one or more spectators must carry large plus momentum into the $x^- = 0$ vertex. Such contributions cancel by the causal identity.

Subtraction and Cancellation. Purely collinear divergences will remain after soft gluon cancellations, and are factorized from the transition amplitude into an “eikonal” distribution function [19] for partons collinear

to β . We shall assume that this factorization can be implemented by collinear subtractions [1, 4].

Following the inductive approach described above, we consider integrals over minus momenta q_i^- of “active” loops, connecting the active eikonal line and the β eikonal in (12) at fixed values of their plus momenta. Any double-counted regions have smaller eikonal field subdiagrams, $\mathcal{E}^{(f')}$, and are assumed finite. Loops connecting the β eikonal with spectator lines are kept fixed in the eikonal field, with minus momenta at order m^2/Q .

Now when a subset of active plus loop momenta flowing between the β eikonal and the active loops has $q^+ \sim Q$, any soft gluons factorize from the resulting larger β jet in the normal fashion [1, 4, 17]. An example is gluon k of Figs. 1a, b and c when $q^+ \sim Q$. These contributions are removed by subtractions for the β eikonal jet. This leaves only the case when all active loops have plus momenta of order m or less, and in this case the minus momenta of these loops can be deformed to order m or more from the origin, away from poles of the β eikonal lines. But then spectator exchanges with minus momenta at order m^2/Q are suppressed, unless they flow into the β eikonal before the earliest active-exchange loop. For example, with $q^+ \leq m$ in Fig. 1, Figs. 1b and 1c are suppressed, while only 1a remains leading in this region. Indeed, any diagram in which a spectator exchange connects to the β eikonal between an active exchange and the vertex at $x^- = 0$ vanishes up to a power of Q after integration of the active-exchange minus momentum [16].

This reasoning results in a decoupling of spectator from active exchange, so that we may represent the subtracted form of \mathcal{M} as

$$[\mathcal{M}(p_a, \{p_t\})]^{(\text{Sub})} = \left\langle 0 \left| \Phi^{(f')}(0, -\infty) \bar{C}_{f'f}(p_a) \right. \right. \\ \left. \times \prod_t \int d^2 \mathbf{x}_t e^{i \mathbf{p}_{t,\perp} \cdot \mathbf{x}_t} W_-^{(t)}(\mathbf{x}_t) \right| 0 \rangle \Psi_f(p_a, \{p^-_t, \mathbf{x}_t\}), \\ \bar{C}_{f'f} = \left\langle 0 \left| \Phi^{(f')}(0, -\infty) C_{f'f} \Phi^{(f)}(0, -\infty) \right| 0 \right\rangle^{(\text{Sub})}, \quad (13)$$

where sums over color indices are implicit, and the superscript “(Sub)” denotes subtractions. The collinear subtractions for each color projection of $\bar{C}_{f'f}$ are equivalent to dividing the timelike form factor for that color projection by its spacelike version, a result that follows from the exponentiation properties of the eikonal form factors [20].

We now close the plus integration contours that flow into spectators in the hybrid form, Eq. (9), for Eq. (13) in the lower half-plane, picking up only final-state poles. The resulting leading regions of \mathcal{M} are characterized either by enlarged β jets, which are eliminated by subtractions, or by multiple subdiagrams $w_i(q_i)$ that carry vanishing net plus momentum q_i^+ into the hard vertex at $x^- = 0$. The simplest of the $w_i(q_i)$ are single exchanged gluons, but more generally, they are generalized *webs*, defined by extending the concept identified for form factors

[20] as follows. A subdiagram exchanged between the β eikonal and the active eikonal is a web when it is irreducible under cutting of the two eikonal lines. Similarly, when a subdiagram is exchanged between the β eikonal and a spectator eikonal, it is a web if it is irreducible when all connections to the spectator are in the final state, corresponding to the convention that all spectator poles are taken in the lower half plane.

The cancellation of the eikonal field in Eq. (13) is now relatively straightforward. The contributions of $\mathcal{E}^{(f')}$, associated with the spectator-exchange webs w_i , cancel by moving the (real) webs one by one from the amplitude in (13) to the complex conjugate, as in the eikonal identity above. The cancellation of final state interactions ensures that color factors remain the same in each term. Once the spectator-exchange webs have been eliminated, the factors $\bar{C}_{f'f}$ and their complex conjugates are neighboring factors in color traces. The subtracted eikonal form factor $\bar{C}_{f'f}$ of Eq. (13) for each color representation is a pure phase. These phases then cancel, because, as noted above, the color representation for $\bar{C}_{f'f}$ is the same in the amplitude and the complex conjugate after the cancellation of final state interactions.

As a simple example, consider $\mathcal{M}(p_a)$ in an abelian theory with coupling α . Taking for simplicity a single spectator of momentum p_s , with unit charge and with a neutral initial incoming state, we find from Eq. (13),

$$[\mathcal{M}(p_s^-, b)]^{(\text{Sub})} \propto \Psi_f(p_s^-, \mathbf{b}) \times \exp \left(i \frac{\alpha}{2\pi} \int \frac{d^2 q_\perp}{q_\perp^2} [e^{i\mathbf{b} \cdot \mathbf{q}_\perp} - 1] \right), \quad (14)$$

in which the phase is explicit.

Weights, Transverse Scales and Phases. Our discussion above assumes a single scale m for transverse momenta, which sets the off-shellness of virtual lines at order m^2 . For subdiagrams whose lines have, for example, much lower transverse momenta, $m' \ll m$, lines at order m may be considered as part of the hard scattering. By the power counting of Refs. [10, 15], we can order sets of spectators according to their transverse momenta, and, in effect, can start with lines of the lowest transverse momenta and “work our way in” to the hard scattering. Softer, non-collinear lines can attach ladders of the p jet with differing transverse momenta only to final state partons and to the β eikonal, and not other ladders. Such final state soft gluon connections cancel in sums over fi-

nal states. Eikonal field exchange gluons are treated as above for each scale m .

Our results on soft-gluon cancellation depend crucially on an inclusive sum over final states. Only after the cancellation of final states can we move spectator-connected webs across the final state, canceling the phases that they generate in Eq. (12). This cancellation will persist only to scales where soft radiation emitted from spectator lines into the final state is fully summed over. For fixed transverse scale m , wide-angle final state radiation from spectators is characterized by energies of order m^2/Q , with Q the spectator momentum. So long as the weight $S(N)$ is chosen so that radiation of this scale is summed inclusively *in all directions*, soft logarithms will cancel in the transition probability. Such weighted \mathcal{S} are thus infrared finite, and correspondingly, cross sections defined with these weights factorize. At the same time, if a weighted cross section does not allow radiation greater than energy E_0 in *any* angular region at wide angles, we may anticipate the noncancellation of radiation at that energy scale, leading to logarithms of the ratio E_0/Q at each order, whether from spectators, or from high- p_T partons in directions allowed by $S(N)$ [21]. At sufficiently high orders, the effects of phases due to emission from spectators of transverse momentum $\sqrt{E_0 Q}$ or greater also begin to appear in finite logarithmic corrections to the cross section, with enhancements associated with the emission of those spectators at ordered transverse momentum [8].

In summary, we have outlined arguments for the cancellation of infrared singularities in transition probabilities with colored incoming lines and final-state jets. We have also seen that the pattern of cancellation is related to the generation of phases in hard-scattering amplitudes, the exchange of Glauber gluons, and the flow of energy into final states. Further investigation should shed more light on these important issues, and their implications for phenomenology.

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